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Application of intuitionistic neutrosophic graph structures in decision-making

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Abstract. In this research study, we present concept of intuitionistic neutrosophic graph structures. We introduce the certain operations on intuitionistic neutrosophic graph structures and elaborate them with suitable examples. Further, we investigate some remarkable properties of these operators. Moreover, we discuss a highly worthwhile real-life application of intuitionistic neutrosophic graph structures in decision-making. Lastly, we elaborate general procedure of our application by designing an algorithm.

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1. Introduction

Graphical models are extensively useful tools for solving combinatorial problems of different fields including optimization, algebra, computer science, topology and operations research etc. Fuzzy graphical models are comparatively more close to nature, because in nature vagueness and ambiguity occurs. There are many complex phenomena and processes in science and technology having incomplete information. To deal such cases we needed a theory different from classical mathematics. Graph structures as generalized simple graphs are widely used for study of edge colored and edge signed graphs, also helpful and copiously used for studying large domains of computer science. Initially in 1965, Zadeh [29] proposed the notion of fuzzy sets to handle uncertainty in a lot of real applications. Fuzzy set theory is finding large number of applications in real time systems, where information inherent in systems has various levels of precision. Afterwards, Turksen [26] proposed the idea of interval-valued fuzzy set. But in various systems, there are membership and non-membership values, membership value is in favor of an event and non-membership value is against of that event. Atanassov [8] proposed the notion of intuitionistic

fuzzy set in 1986. The intuitionistic fuzzy sets are more practical and applicable in real-life situations. Intuitionistic fuzzy set deal with incomplete information, that is, degree of membership function, non-membership function but not indeterminate and inconsistent information that exists definitely in many systems, including belief system, decision-support systems etc. In 1998, Smarandache [24] proposed another notion of imprecise data named as neutrosophic sets. "Neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra". Neutrosophic set is recently proposed powerful formal framework. For convenient usage of neutrosophic sets in real-life situations, Wang et al. [27] proposed single-valued neutrosophic set as a generalization of intuitionistic fuzzy set[8]. A neutrosophic set has three independent components having values in unit interval [0, 1]. On the other hand, Bhowmik and Pal [10, 11] introduced the notions of intuitionistic neutrosophic sets and relations. Kauffman [16] defined fuzzy graph on the basis of Zadeh's fuzzy relations [30]. Rosenfeld [21] investigated fuzzy analogue of various graph-theoretic ideas in 1975. Later on, Bhattacharya gave some remarks on fuzzy graph in 1987. Bhutani and Rosenfeld discussed M-strong fuzzy graphs with their properties in [12]. In 2011, Dinesh and Ramakrishnan [15] put forward fuzzy graph structures and investigated its properties. In 2016, Akram and Akmal [1] proposed the notion of bipolar fuzzy graph structures. Broumi et al. [13] portrayed single-valued neutrosophic graphs. Akram and Shahzadi [2] introduced the notion of neutrosophic soft graphs with applications. Akram and Shahzadi [4] highlighted some flaws in the definitions of Broumi et al. [13] and Shah-Hussain [22]. Akram et al. [5] also introduced the single-valued neutrosophic hypergraphs. Representation of graphs using intuitionistic neutrosophic soft sets was discussed in [3]. In this paper, we present concept of intuitionistic neutrosophic graph structures. We introduce the certain operations on intuitionistic neutrosophic graph structures and elaborate them with suitable examples. Further, we investigate some remarkable properties of these operators. Moreover, we discuss a highly worthwhile real-life application of intuitionistic neutrosophic graph structures in decision-making. Lastly, we elaborate general procedure of our application by designing an algorithm.
We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [3,6, 7, 9, 13, 14, 17, 18, 20, 22, 23, 25, 28, 30].

2. Intuitionistic Neutrosophic Graph Structures

Definition 2.1. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \ldots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, Cartesian product of $\check{G}_1$ and $\check{G}_1$ is defined as:

$$\check{G}_1 \times \check{G}_2 = (P \times P', P_1 \times P'_1, P_2 \times P'_2, \ldots, P_r \times P'_r),$$

where $P_h \times P'_h = \{(k_1 l)(k_2 l) \mid l \in P',\ k_1k_2 \in P_h\} \cup \{(kl_1)(kl_2) \mid k \in p, l_1l_2 \in P'_h\}$, $h = (1, 2, \ldots, r)$.

Definition 2.2. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \ldots, P_n)$ and $\check{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, cross product of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 * \check{G}_2 = (P * P', P_1 * P'_1, P_2 * P'_2, \ldots, P_r * P'_r),$$

where $P_h * P'_h = \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h,\, l_1l_2 \in P'_h\}$, $h = (1, 2, \ldots, r)$.

Definition 2.3. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \ldots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, lexicographic product of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 \bullet \check{G}_2 = (P \bullet P', P_1 \bullet P'_1, P_2 \bullet P'_2, \ldots, P_r \bullet P'_r),$$

where $P_h \bullet P'_h = \{(kl_1)(kl_2) \mid k \in P,\, l_1l_2 \in P'_h\} \cup \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h, l_1l_2 \in P'_h\}$, $h = (1, 2, \ldots, r)$.

Definition 2.4. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \ldots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, strong product of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 \boxtimes \check{G}_2 = (P \boxtimes P', P_1 \boxtimes P'_1, P_2 \boxtimes P'_2, \ldots, P_r \boxtimes P'_r),$$

where $P_h \boxtimes P'_h = \{(k_1l)(k_2l) \mid l \in P',\, k_1k_2 \in P_h\} \cup \{(kl_1)(kl_2) \mid k \in P, l_1l_2 \in P'_h\} \cup \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h,\, l_1l_2 \in P'_h\}$, $h = (1, 2, \ldots, r)$.

Definition 2.5. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \ldots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \ldots, P'_n)$ be two GSs, composition of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 \circ \check{G}_2 = (P \circ P', P_1 \circ P'_1, P_2 \circ P'_2, \ldots, P_r \circ P'_r),$$

where $P_h \circ P'_h = \{(k_1l)(k_2l) \mid l \in P',\, k_1k_2 \in P_h\} \cup \{(kl_1)(kl_2) \mid k \in P, l_1l_2 \in P'_h\} \cup \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h,\, l_1, l_2 \in P'$ such that $l_1 \neq l_2\}$, $h = (1, 2, \ldots, r)$.

Definition 2.6. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \ldots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, union of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 \cup \check{G}_2 = (P \cup P', P_1 \cup P'_1, P_2 \cup P'_2, \ldots, P_r \cup P'_r).$$

Definition 2.7. ([23]). Let $\check{G}_1 = (P, P_1, P_2, \ldots, P_r)$ and $\check{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, join of $\check{G}_1$ and $\check{G}_2$ is defined as:

$$\check{G}_1 + \check{G}_2 = (P + P', P_1 + P'_1, P_2 + P'_2, \ldots, P_r + P'_r),$$

where $P + P' = P \cup P'$, $P_h + P'_h = P_h \cup P'_h \cup P''_h$ for $h = (1, 2, \ldots, r)$. P''_h consists of all those edges which join the vertices of P and P'.

Definition 2.8. ([19]). Let V be a fixed set. A generalized intuitionistic fuzzy set I of V is an object having the form $I=\{(v, \mu_I(v), \nu_I(v)) | v \in V\}$, where the functions $\mu_I : V \to [0, 1]$ and $\nu_I : V \to [0, 1]$ define the degree of membership and degree of nonmembership of an element $v \in V$, respectively, such that

$$\min\{\mu_I(v), \nu_I(v)\} \leq 0.5, \text{ for all } v \in V.$$

This condition is called the generalized intuitionistic condition.

Definition 2.9. ([10, 11]). A set $I = \{T_I(v), I_I(v), F_I(v) : v \in V\}$ is said to be an intuitionistic neutrosophic (IN)set, if

(i) $\{T_I(v) \wedge I_I(v)\} \leq 0.5$, $\{I_I(v) \wedge F_I(v)\} \leq 0.5$, $\{F_I(v) \wedge T_I(v)\} \leq 0.5$,
(ii) $0 \leq T_I(v) + I_I(v) + F_I(v) \leq 2$.

Definition 2.10. An intuitionistic neutrosophic graph is a pair $G = (A, B)$ with underlying set V, where T_A, F_A, $I_A : V \to [0, 1]$ denote the truth, falsity and indeterminacy membership values of the vertices in V and T_B, F_B, $I_B : E \subseteq V \times V \to [0, 1]$ denote the truth, falsity and indeterminacy membership values of the edges $kl \in E$ such that

(i) $T_B(kl) \le T_A(k) \wedge T_A(l)$, $F_B(kl) \le F_A(k) \vee F_A(l)$, $I_B(kl) \le I_A(k) \wedge I_A(l)$,
(ii) $T_B(kl) \wedge I_B(kl) \le 0.5$, $T_B(kl) \wedge F_B(kl) \le 0.5$, $I_B(kl) \wedge F_B(lk) \le 0.5$,
(iii) $0 \le T_B(kl) + F_B(kl) + I_B(kl) \le 2$, $\forall\ k, l \in V$.

Definition 2.11. $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ is said to be an intuitionistic neutrosophic graph structure(INGS) of graph structure $\check{G} = (P, P_1, P_2, \ldots, P_r)$, if $O = < k, T(k), I(k), F(k) >$ and $O_h = < kl, T_h(kl), I_h(kl), F_h(kl) >$ are the intuitionistic neutrosophic(IN) sets on the sets P and P_h, respectively such that
(i) $T_h(kl) \le T(k) \wedge T(l)$, $I_h(kl) \le I(k) \wedge I(l)$, $F_h(kl) \le F(k) \vee F(l)$,
(ii) $T_h(kl) \wedge I_h(kl) \le 0.5$, $T_h(kl) \wedge F_h(kl) \le 0.5$, $I_h(kl) \wedge F_h(kl) \le 0.5$,
(iii) $0 \le T_h(kl) + I_h(kl) + F_h(kl) \le 2$, $for\ all\ kl \in O_h,\ h \in \{1, 2, \ldots, r\}$,
where, O and O_h are underlying vertex and h-edge sets of INGS $\check{G}_i$, $h \in \{1, 2, \ldots, r\}$.

Example 2.12. An intuitionistic neutrosophic graph structure is represented in Fig. 1.

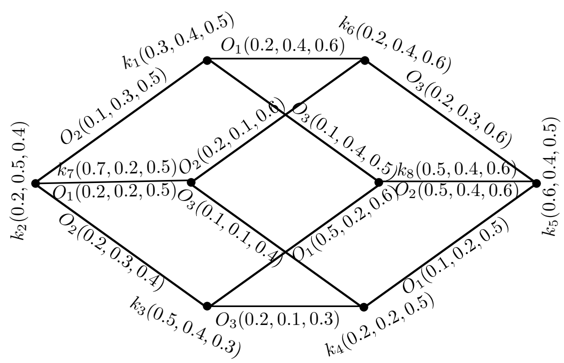


FIGURE 1. An intuitionistic neutrosophic graph structure

Now we define the operations on INGSs.

Definition 2.13. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, \ldots, Q_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively.
Cartesian product of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \times \check{G}_{i2} = (O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}, \ldots, O_{1r} \times O_{2r}),$$

is defined as:

$$\text{(i)} \begin{cases} T_{(O_1 \times O_2)}(kl) = (T_{O_1} \times T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 \times O_2)}(kl) = (I_{O_1} \times I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 \times O_2)}(kl) = (F_{O_1} \times F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$$

for all $kl \in P_1 \times P_2$,

$$\text{(ii)} \begin{cases} T_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \times T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \times I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \times F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \end{cases}$$

for all $k \in P_1$, $l_1l_2 \in P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h}\times O_{2h})}(k_1l)(k_2l) = (T_{O_{1h}} \times T_{O_{2h}})(k_1l)(k_2l) = T_{O_2}(l) \wedge T_{O_{1h}}(k_1k_2) \\ I_{(O_{1h}\times O_{2h})}(k_1l)(k_2l) = (I_{O_{1h}} \times I_{O_{2h}})(k_1l)(k_2l) = I_{O_2}(l) \wedge I_{O_{2h}}(k_1k_2) \\ F_{(O_{1h}\times O_{2h})}(k_1l)(q_2l) = (F_{O_{1h}} \times F_{O_{2h}})(k_1l)(k_2l) = F_{O_2}(l) \vee F_{O_{2h}}(k_1k_2) \end{cases}$
for all $l \in P_2$, $k_1k_2 \in P_{1h}$.

Example 2.14. Consider $\check{G}_{i1} = (O_1, O_{11}, O_{12})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22})$ are two INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12})$ and $\check{G}_2 = (P_2, P_{21}, P_{22})$ respectively, as represented in Fig. 2, where $P_{11} = \{k_1k_2\}$, $P_{12} = \{k_3k_4\}$, $P_{21} = \{l_1l_2\}$, $P_{22} = \{l_2l_3\}$.

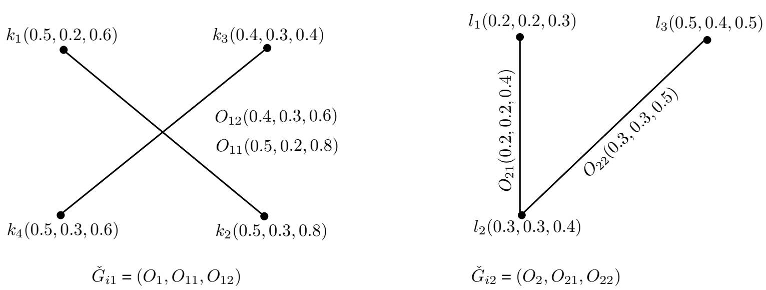
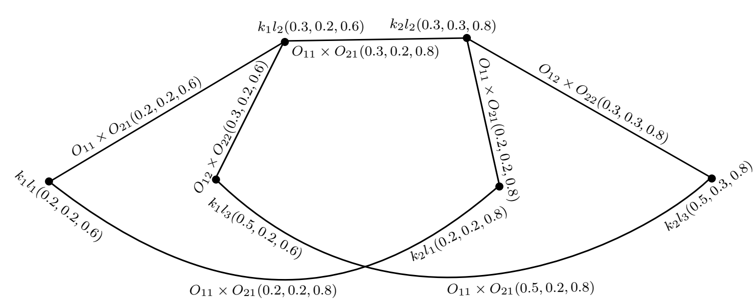


FIGURE 2. Two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$

Cartesian product of $\check{G}_{i1}$ and $\check{G}_{i2}$ defined as $\check{G}_{i1} \times \check{G}_{i2} = \{O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}\}$ is represented in Fig. 3.



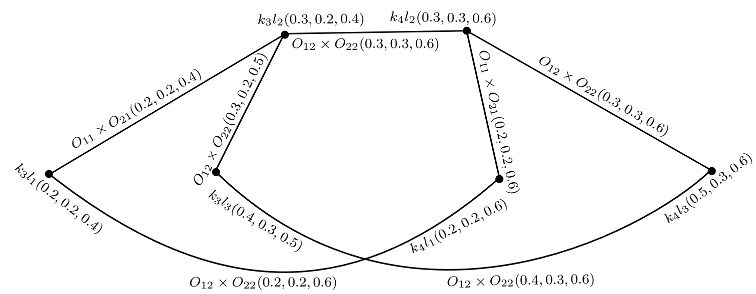


FIGURE 3. $\check{G}_{i1} \times \check{G}_{i2}$

Theorem 2.15. *Cartesian product* $\check{G}_{i1} \times \check{G}_{i2} = (O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}, \ldots, O_{1r} \times O_{2r})$ *of two INGSs of GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1 \times \check{G}_2$.

Proof. We consider two cases:

Case 1: For $k \in P_1$, $l_1l_2 \in P_{2h}$

$$\begin{aligned} T_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ &\leq T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1 \times O_2)}(kl_1) \wedge T_{(O_1 \times O_2)}(kl_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) &= I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ &\leq I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\ &= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)] \\ &= I_{(O_1 \times O_2)}(kl_1) \wedge I_{(O_1 \times O_2)}(kl_2), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) &= F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \\ &\leq F_{O_1}(k) \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\ &= [F_{O_1}(k) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k) \vee F_{O_2}(l_2)] \\ &= F_{(O_1 \times O_2)}(kl_1) \vee F_{(O_1 \times O_2)}(kl_2), \end{aligned}$$

for $kl_1, kl_2 \in P_1 \times P_2$.

Case 2: For $k \in P_2$, $l_1l_2 \in P_{1h}$

$$\begin{aligned} T_{(O_{1h} \times O_{2h})}((l_1k)(l_2k)) &= T_{O_2}(k) \wedge T_{O_{1h}}(l_1l_2) \\ &\leq T_{O_2}(k) \wedge [T_{O_1}(l_1) \wedge T_{O_1}(l_2)] \\ &= [T_{O_2}(k) \wedge T_{O_1}(l_1)] \wedge [T_{O_2}(k) \wedge T_{O_1}(l_2)] \\ &= T_{(O_1 \times O_2)}(l_1k) \wedge T_{(O_1 \times O_2)}(l_2k), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h}\times O_{2h})}((l_1k)(l_2k)) &= I_{O_2}(k)\wedge I_{O_{1h}}(l_1l_2)\\ &\le I_{O_2}(k)\wedge [I_{O_1}(l_1)\wedge I_{O_1}(l_2)]\\ &= [I_{O_2}(k)\wedge I_{O_1}(l_1)]\wedge [I_{O_2}(k)\wedge I_{O_1}(l_2)]\\ &= I_{(O_1\times O_2)}(l_1k)\wedge I_{(O_1\times O_2)}(l_2k), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h}\times O_{2h})}((l_1k)(l_2k)) &= F_{O_2}(k)\vee F_{O_{1h}}(l_1l_2)\\ &\le F_{O_2}(k)\vee [F_{O_1}(l_1)\vee F_{O_1}(l_2)]\\ &= [F_{O_2}(k)\vee F_{O_1}(l_1)]\vee [F_{O_2}(k)\vee F_{O_1}(l_2)]\\ &= F_{(O_1\times O_2)}(l_1k)\vee F_{(O_1\times O_2)}(l_2k), \end{aligned}$$

for $l_1k, l_2k\in P_1\times P_2$.

Both cases exists $\forall h\in\{1,2,\dots,r\}$. This completes the proof. □

Definition 2.16. Let $\check{G}_{i1}=(O_1,O_{11},O_{12},\dots,Q_{1r})$ and $\check{G}_{i2}=(O_2,O_{21},O_{22},\dots,Q_{2r})$ be INGSs of GSs $\check{G}_1=(P_1,P_{11},P_{12},\dots,P_{1r})$ and $\check{G}_2=(P_2,P_{21},P_{22},\dots,P_{2r})$, respectively. Cross product of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1}*\check{G}_{i2}=(O_1*O_2,O_{11}*O_{21},O_{12}*O_{22},\dots,O_{1r}*O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1*O_2)}(kl)=(T_{O_1}*T_{O_2})(kl)=T_{O_1}(k)\wedge T_{O_2}(l)\\ I_{(O_1*O_2)}(kl)=(I_{O_1}*I_{O_2})(kl)=I_{O_1}(k)\wedge I_{O_2}(l)\\ F_{(O_1*O_2)}(kl)=(F_{O_1}*F_{O_2})(kl)=F_{O_1}(k)\vee F_{O_2}(l)\end{cases}$

for all $kl\in P_1\times P_2$,

(ii) $\begin{cases} T_{(O_{1h}*O_{2h})}(k_1l_1)(k_2l_2)=(T_{O_{1h}}*T_{O_{2h}})(k_1l_1)(k_2l_2)=T_{O_{1h}}(k_1k_2)\wedge T_{O_{2h}}(l_1l_2)\\ I_{(O_{1h}*O_{2h})}(k_1l_1)(k_2l_2)=(I_{O_{1h}}*I_{O_{2h}})(k_1l_1)(k_2l_2)=I_{O_{1h}}(k_1k_2)\wedge I_{O_{2h}}(l_1l_2)\\ F_{(O_{1h}*O_{2h})}(k_1l_1)(k_2l_2)=(F_{O_{1h}}*F_{O_{2h}})(k_1l_1)(k_2l_2)=F_{O_{1h}}(k_1k_2)\vee F_{O_{2h}}(l_1l_2)\end{cases}$

for all $k_1k_2\in P_{1h}$, $l_1l_2\in P_{2h}$.

Example 2.17. Cross product of INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as $\check{G}_{i1}*\check{G}_{i2}=\{O_1*O_2,O_{11}*O_{21},O_{12}*O_{22}\}$ and is represented in Fig. 4.

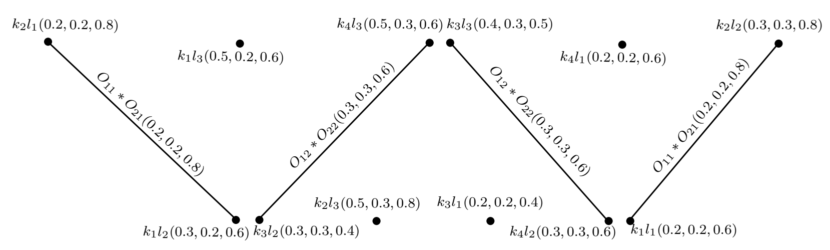


FIGURE 4. $\check{G}_{i1}*\check{G}_{i2}$

Theorem 2.18. *Cross product* $\check{G}_{i1}*\check{G}_{i2}=(O_1*O_2,O_{11}*O_{21},O_{12}*O_{22},\dots,O_{1r}*O_{2r})$ *of two INGSs of GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1*\check{G}_2$.

Proof. For all $k_1l_1, k_2l_2 \in P_1 * P_2$

$$\begin{aligned} T_{(O_{1h}*O_{2h})}((k_1l_1)(k_2l_2)) &= T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2) \\ &\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k_1) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k_2) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1*O_2)}(k_1l_1) \wedge T_{(O_1*O_2)}(k_2l_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h}*O_{2h})}((k_1l_1)(k_2l_2)) &= I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}l_1l_2) \\ &\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\ &= [I_{O_1}(k_1) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k_2) \wedge I_{O_2}(l_2)] \\ &= I_{(O_1*O_2)}(k_1l_1) \wedge I_{(O_1*O_2)}(k_2l_2), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h}*O_{2h})}((k_1l_1)(k_2l_2)) &= F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2) \\ &\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\ &= [F_{O_1}(k_1) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k_2) \vee F_{O_2}(l_2)] \\ &= F_{(O_1*O_2)}(k_1l_1) \vee F_{(O_1*O_2)}(k_2l_2), \end{aligned}$$

for $h \in \{1, 2, \ldots, r\}$. This completes the proof. □

Definition 2.19. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. Lexicographic product of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \bullet \check{G}_{i2} = (O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}, \ldots, O_{1r} \bullet O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1\bullet O_2)}(kl) = (T_{O_1} \bullet T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1\bullet O_2)}(kl) = (I_{O_1} \bullet I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1\bullet O_2)}(kl) = (F_{O_1} \bullet F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$
for all $kl \in P_1 \times P_2$

(ii) $\begin{cases} T_{(O_{1h}\bullet O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \bullet T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h}\bullet O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \bullet I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h}\bullet O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \bullet F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \end{cases}$
for all $k \in P_1$, $l_1l_2 \in P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h}\bullet O_{2h})}(k_1l_1)(k_2l_2) = (T_{O_{1h}} \bullet T_{O_{2h}})(k_1l_1)(k_2l_2) = T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h}\bullet O_{2h})}(k_1l_1)(k_2l_2) = (I_{O_{1h}} \bullet I_{O_{2h}})(k_1l_1)(k_2l_2) = I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h}\bullet O_{2h})}(k_1l_1)(k_2l_2) = (F_{O_{1h}} \bullet F_{O_{2h}})(k_1l_1)(k_2l_2) = F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2) \end{cases}$
for all $k_1k_2 \in P_{1h}$, $l_1l_2 \in P_{2h}$.

Example 2.20. Lexicographic product of INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as $\check{G}_{i1} \bullet \check{G}_{i2} = \{O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}\}$ and is represented in Fig. 5.

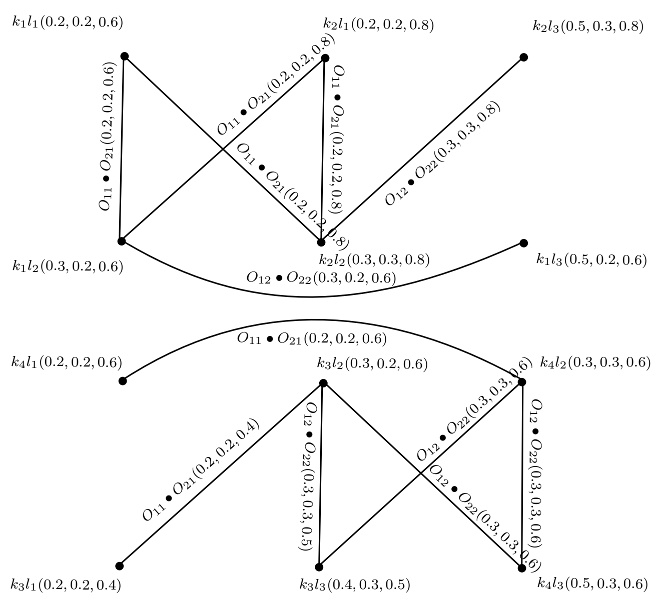


FIGURE 5. $\check{G}_{i1}\bullet\check{G}_{i2}$

Theorem 2.21. *Lexicographic product* $\check{G}_{i1}\bullet\check{G}_{i2}=(O_1\bullet O_2, O_{11}\bullet O_{21}, O_{12}\bullet O_{22},\ldots,O_{1r}\bullet O_{2r})$ *of two INGSs of the GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1\bullet\check{G}_2$.

Proof. We consider two cases:

Case 1: For $k\in P_1$, $l_1l_2\in P_{2h}$

$$\begin{aligned}T_{(O_{1h}\bullet O_{2h})}((kl_1)(kl_2))&=T_{O_1}(k)\wedge T_{O_{2h}}(l_1l_2)\\&\leq T_{O_1}(k)\wedge[T_{O_2}(l_1)\wedge T_{O_2}(l_2)]\\&=[T_{O_1}(k)\wedge T_{O_2}(l_1)]\wedge[T_{O_1}(k)\wedge T_{O_2}(l_2)]\\&=T_{(O_1\bullet O_2)}(kl_1)\wedge T_{(O_1\bullet O_2)}(kl_2),\end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\bullet O_{2i})}((kl_1)(kl_2))&=I_{O_1}(k)\wedge I_{O_{2h}}(l_1l_2)\\&\leq I_{O_1}(k)\wedge[I_{O_2}(l_1)\wedge I_{O_2}(l_2)]\\&=[I_{O_1}(k)\wedge I_{O_2}(l_1)]\wedge[I_{O_1}(k)\wedge I_{O_2}(l_2)]\\&=I_{(O_1\bullet O_2)}(kl_1)\wedge I_{(O_1\bullet O_2)}(kl_2),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\bullet O_{2i})}((kl_1)(kl_2))&=F_{O_1}(k)\vee F_{O_{2h}}(l_1l_2)\\&\leq F_{O_1}(k)\vee[F_{O_2}(l_1)\vee F_{O_2}(l_2)]\\&=[F_{O_1}(k)\vee F_{O_2}(l_1)]\vee[F_{O_1}(k)\vee F_{O_2}(l_2)]\\&=F_{(O_1\bullet O_2)}(kl_1)\vee F_{(O_1\bullet O_2)}(kl_2),\end{aligned}$$

for $kl_1,kl_2\in P_1\bullet P_2$.

Case 2: For $k_1k_2 \in P_{1h}, l_1l_2 \in P_{2h}$

$$\begin{aligned}
T_{(O_{1h}\bullet O_{2h})}((k_1l_1)(k_2l_2)) &= T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2)\\
&\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)]\\
&= [T_{O_1}(k_1) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k_2) \wedge T_{O_2}(l_2)]\\
&= T_{(O_1\bullet O_2)}(k_1l_1) \wedge T_{(O_1\bullet O_2)}(k_2l_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h}\bullet O_{2h})}((k_1l_1)(k_2l_2)) &= I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2)\\
&\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)]\\
&= [I_{O_1}(k_1) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k_2) \wedge I_{O_2}(l_2)]\\
&= I_{(O_1\bullet O_2)}(k_1l_1) \wedge I_{(O_1\bullet O_2)}(k_2l_2),
\end{aligned}$$

$$\begin{aligned}
F_{(O_{1h}\bullet O_{2h})}((k_1l_1)(k_2l_2)) &= F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2)\\
&\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)]\\
&= [F_{O_1}(k_1) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k_2) \vee F_{O_2}(l_2)]\\
&= F_{(O_1\bullet O_2)}(k_1l_1) \vee F_{(O_1\bullet O_2)}(k_2l_2),
\end{aligned}$$

for $k_1l_1, k_2l_2 \in P_1 \bullet P_2$.
Both cases hold for $h \in \{1, 2, \ldots, r\}$. This completes the proof.

□

Definition 2.22. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, \ldots, O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. Strong product of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \boxtimes \check{G}_{i2} = (O_1 \boxtimes O_2, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}, \ldots, O_{1r} \boxtimes O_{2r}),$$

is defined as:

$$\text{(i)}\ \begin{cases} T_{(O_1\boxtimes O_2)}(kl) = (T_{O_1} \boxtimes T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l)\\ I_{(O_1\boxtimes O_2)}(kl) = (I_{O_1} \boxtimes I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l)\\ F_{(O_1\boxtimes O_2)}(kl) = (F_{O_1} \boxtimes F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$$

for all $kl \in P_1 \times P_2$,

$$\text{(ii)}\ \begin{cases} T_{(O_{1h}\boxtimes O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \boxtimes T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2)\\ I_{(O_{1h}\boxtimes O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \boxtimes I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2)\\ F_{(O_{1h}\boxtimes O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \boxtimes F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \end{cases}$$

for all $k \in P_1$, $l_1l_2 \in P_{2h}$,

$$\text{(iii)}\ \begin{cases} T_{(O_{1h}\boxtimes O_{2h})}(k_1l)(k_2l) = (T_{O_{1h}} \boxtimes T_{O_{2h}})(k_1l)(k_2l) = T_{O_2}(l) \wedge T_{O_{1h}}(k_1k_2)\\ I_{(O_{1h}\boxtimes O_{2h})}(k_1l)(k_2l) = (I_{O_{1h}} \boxtimes I_{O_{2h}})(k_1l)(k_2l) = I_{O_2}(l) \wedge I_{O_{2h}}(k_1k_2)\\ F_{(O_{1h}\boxtimes O_{2h})}(k_1l)(k_2l) = (F_{O_{1h}} \boxtimes F_{O_{2h}})(k_1l)(k_2l) = F_{O_2}(l) \vee F_{O_{2h}}(k_1k_2) \end{cases}$$

for all $l \in P_2$, $k_1k_2 \in P_{1h}$,

$$\text{(iv)}\ \begin{cases} T_{(O_{1h}\boxtimes O_{2h})}(k_1l_1)(k_2l_2) = (T_{O_{1h}} \boxtimes T_{O_{2h}})(k_1l_1)(k_2l_2) = T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2)\\ I_{(O_{1h}\boxtimes O_{2h})}(k_1l_1)(k_2l_2) = (I_{O_{1h}} \boxtimes I_{O_{2h}})(k_1l_1)(k_2l_2) = I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2)\\ F_{(O_{1h}\boxtimes O_{2h})}(k_1l_1)(k_2l_2) = (F_{O_{1h}} \boxtimes F_{O_{2h}})(k_1l_1)(k_2l_2) = F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2) \end{cases}$$

for all $k_1k_2 \in P_{1h}$, $l_1l_2 \in P_{2h}$.

Example 2.23. Strong product of INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as $\check{G}_{i1} \boxtimes \check{G}_{i2} = \{O_1 \boxtimes O_2, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}\}$ and is represented in Fig. 6.

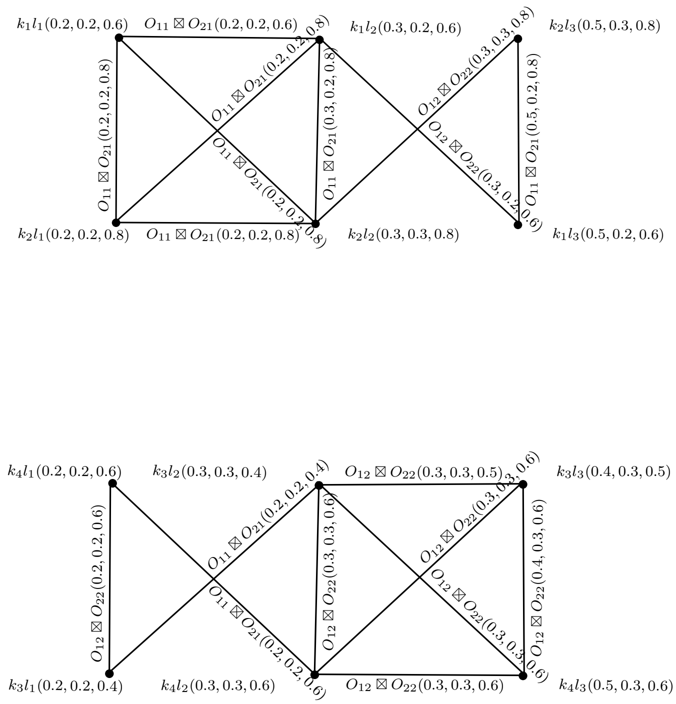


FIGURE 6. $\check{G}_{i1}\boxtimes\check{G}_{i2}$

Theorem 2.24. *Strong product* $\check{G}_{i1}\boxtimes\check{G}_{i2}=(O_1\boxtimes O_2, O_{11}\boxtimes O_{21}, O_{12}\boxtimes O_{22},\ldots,O_{1r}\boxtimes O_{2r})$ *of two INGSs of the GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1\boxtimes\check{G}_2$.

Proof. There are three cases:

Case 1: For $k\in P_1$, $l_1l_2\in P_{2h}$

$$\begin{aligned}
T_{(O_{1h}\boxtimes O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k)\wedge T_{O_{2h}}(l_1l_2)\\
&\leq T_{O_1}(k)\wedge[T_{O_2}(l_1)\wedge T_{O_2}(l_2)]\\
&= [T_{O_1}(k)\wedge T_{O_2}(l_1)]\wedge[T_{O_1}(k)\wedge T_{O_2}(l_2)]\\
&= T_{(O_1\boxtimes O_2)}(kl_1)\wedge T_{(O_1\boxtimes O_2)}(kl_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h}\boxtimes O_{2h})}((kl_1)(kl_2)) &= I_{O_1}(k)\wedge I_{O_{2h}}(l_1l_2)\\
&\leq I_{O_1}(k)\wedge[I_{O_2}(l_1)\wedge I_{O_2}(l_2)]\\
&= [I_{O_1}(k)\wedge I_{O_2}(l_1)]\wedge[I_{O_1}(k)\wedge I_{O_2}(l_2)]\\
&= I_{(O_1\boxtimes O_2)}(kl_1)\wedge I_{(O_1\boxtimes O_2)}(kl_2),
\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\boxtimes O_{2h})}((kl_1)(kl_2)) &= F_{O_1}(k)\vee F_{O_{2h}}(l_1l_2)\\ &\le F_{O_1}(k)\vee[F_{O_2}(l_1)\vee F_{O_2}(l_2)]\\ &= [F_{O_1}(k)\vee F_{O_2}(l_1)]\vee[F_{O_1}(k)\vee F_{O_2}(l_2)]\\ &= F_{(O_1\boxtimes O_2)}(kl_1)\vee F_{(O_1\boxtimes O_2)}(kl_2),\end{aligned}$$

for $kl_1, kl_2 \in P_1 \boxtimes P_2$.

Case 2: For $k\in P_2$, $l_1l_2\in P_{1h}$

$$\begin{aligned}T_{(O_{1h}\boxtimes O_{2h})}((l_1k)(l_2k)) &= T_{O_2}(k)\wedge T_{O_{1h}}(l_1l_2)\\ &\le T_{O_2}(k)\wedge[T_{O_1}(l_1)\wedge T_{O_1}(l_2)]\\ &= [T_{O_2}(k)\wedge T_{O_1}(l_1)]\wedge[T_{O_2}(k)\wedge T_{O_1}(l_2)]\\ &= T_{(O_1\boxtimes O_2)}(l_1k)\wedge T_{(O_1\boxtimes O_2)}(l_2k),\end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\boxtimes O_{2h})}((l_1k)(l_2k)) &= I_{O_2}(k)\wedge I_{O_{1h}}(l_1l_2)\\ &\le I_{O_2}(k)\wedge[I_{O_1}(l_1)\wedge I_{O_1}(l_2)]\\ &= [I_{O_2}(k)\wedge I_{O_1}(l_1)]\wedge[I_{O_2}(k)\wedge I_{O_1}(l_2)]\\ &= I_{(O_1\boxtimes O_2)}(l_1k)\wedge I_{(O_1\boxtimes O_2)}(l_2k),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\boxtimes O_{2h})}((l_1k)(l_2k)) &= F_{O_2}(k)\vee F_{O_{1h}}(l_1l_2)\\ &\le F_{O_2}(k)\vee[F_{O_1}(l_1)\vee F_{O_1}(l_2)]\\ &= [F_{O_2}(k)\vee F_{O_1}(l_1)]\vee[F_{O_2}(k)\vee F_{O_1}(l_2)]\\ &= F_{(O_1\boxtimes O_2)}(l_1k)\vee F_{(O_1\boxtimes O_2)}(l_2k),\end{aligned}$$

for $l_1k, l_2k \in P_1 \boxtimes P_2$.

Case 3: For every $k_1k_2 \in P_{1h}, l_1l_2 \in P_{2h}$

$$\begin{aligned}T_{(O_{1h}\boxtimes O_{2h})}((k_1l_1)(k_2l_2)) &= T_{O_{1h}}(k_1k_2)\wedge T_{O_{2h}}(l_1l_2)\\ &\le [T_{O_1}(k_1)\wedge T_{O_1}(k_2)]\wedge[T_{O_2}(l_1)\wedge T_{O_2}(l_2)]\\ &= [T_{O_1}(k_1)\wedge T_{O_2}(l_1)]\wedge[T_{O_1}(k_2)\wedge T_{O_2}(l_2)]\\ &= T_{(O_1\boxtimes O_2)}(k_1l_1)\wedge T_{(O_1\boxtimes O_2)}(k_2l_2),\end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\boxtimes O_{2h})}((k_1l_1)(k_2l_2)) &= I_{O_{1h}}(k_1k_2)\wedge I_{O_{2h}}(l_1l_2)\\ &\le [I_{O_1}(k_1)\wedge I_{O_1}(k_2)]\wedge[I_{O_2}(l_1)\wedge I_{O_2}(l_2)]\\ &= [I_{O_1}(k_1)\wedge I_{O_2}(l_1)]\wedge[I_{O_1}(k_2)\wedge I_{O_2}(l_2)]\\ &= I_{(O_1\boxtimes O_2)}(k_1l_1)\wedge I_{(O_1\boxtimes O_2)}(k_2l_2),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\boxtimes O_{2h})}((k_1l_1)(k_2l_2)) &= F_{O_{1h}}(k_1k_2)\vee F_{O_{2h}}(l_1l_2)\\ &\le [F_{O_1}(k_1)\vee F_{O_1}(k_2)]\vee[F_{O_2}(l_1)\vee F_{O_2}(l_2)]\\ &= [F_{O_1}(k_1)\vee F_{O_2}(l_1)]\vee[F_{O_1}(k_2)\vee F_{O_2}(l_2)]\\ &= F_{(O_1\boxtimes O_2)}(k_1l_1)\vee F_{(O_1\boxtimes O_2)}(k_2l_2),\end{aligned}$$

for $k_1l_1, k_2l_2 \in P_1 \boxtimes P_2$, and $h = 1, 2, \ldots, r$.

This completes the proof. □

Definition 2.25. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. The composition of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \circ \check{G}_{i2} = (O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}, \ldots, O_{1r} \circ O_{2r}),$$

is defined as:

$$\text{(i)} \begin{cases} T_{(O_1 \circ O_2)}(kl) = (T_{O_1} \circ T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 \circ O_2)}(kl) = (I_{O_1} \circ I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 \circ O_2)}(kl) = (F_{O_1} \circ F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$$

for all $kl \in P_1 \times P_2$,

$$\text{(ii)} \begin{cases} T_{(O_{1h} \circ O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \circ T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1 l_2) \\ I_{(O_{1h} \circ O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \circ I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1 l_2) \\ F_{(O_{1h} \circ O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \circ F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1 l_2) \end{cases}$$

for all $k \in P_1$, $l_1 l_2 \in P_{2h}$,

$$\text{(iii)} \begin{cases} T_{(O_{1h} \circ O_{2h})}(k_1 l)(k_2 l) = (T_{O_{1h}} \circ T_{O_{2h}})(k_1 l)(k_2 l) = T_{O_2}(l) \wedge T_{O_{1h}}(k_1 k_2) \\ I_{(O_{1h} \circ O_{2h})}(k_1 l)(k_2 l) = (I_{O_{1h}} \circ I_{O_{2h}})(k_1 l)(k_2 l) = I_{O_2}(l) \wedge I_{O_{2h}}(k_1 k_2) \\ F_{(O_{1h} \circ O_{2h})}(k_1 l)(k_2 l) = (F_{O_{1h}} \circ F_{O_{2h}})(k_1 l)(k_2 l) = F_{O_2}(l) \vee F_{O_{2h}}(k_1 k_2) \end{cases}$$

for all $l \in P_2$, $k_1 k_2 \in P_{1h}$,

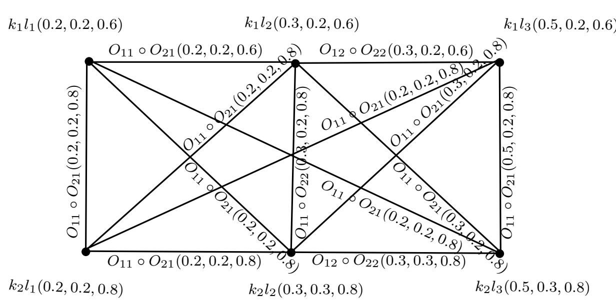
$$\text{(iv)} \begin{cases} T_{(O_{1h} \circ O_{2h})}(k_1 l_1)(k_2 l_2) = (T_{O_{1h}} \circ T_{O_{2h}})(k_1 l_1)(k_2 l_2) = T_{O_{1h}}(k_1 k_2) \wedge T_{O_2}(l_1) \wedge T_{O_2}(l_2) \\ I_{(O_{1h} \circ O_{2h})}(k_1 l_1)(k_2 l_2) = (I_{O_{1h}} \circ I_{O_{2h}})(k_1 l_1)(k_2 l_2) = I_{O_{1h}}(k_1 k_2) \wedge I_{O_2}(l_1) \wedge I_{O_2}(l_2) \\ F_{(O_{1h} \circ O_{2h})}(k_1 l_1)(k_2 l_2) = (F_{O_{1h}} \circ F_{O_{2h}})(k_1 l_1)(k_2 l_2) = F_{O_{1h}}(k_1 k_2) \vee F_{O_2}(l_1) \vee F_{O_2}(l_2) \end{cases}$$

for all $k_1 k_2 \in P_{1h}$, $l_1 l_2 \in P_{2h}$ such that $l_1 \neq l_2$.

Example 2.26. The composition of INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as:

$$\check{G}_{i1} \circ \check{G}_{i2} = \{O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}\}$$

and is represented in Fig. 7.



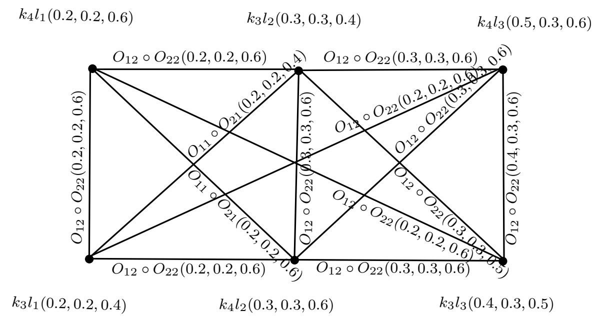


Figure 7. $\check{G}_{i1} \circ \check{G}_{i2}$

Theorem 2.27. *The composition $\check{G}_{i1} \circ \check{G}_{i2} = (O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}, \ldots, O_{1r} \circ O_{2r})$ of two INGSs of GSs $\check{G}_1$ and $\check{G}_2$ is an INGS of $\check{G}_1 \circ \check{G}_2$.*

Proof. We consider three cases:

Case 1: For $k \in P_1$, $l_1l_2 \in P_{2h}$

$$\begin{aligned} T_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ &\leq T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1 \circ O_2)}(kl_1) \wedge T_{(O_1 \circ O_2)}(kl_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) &= I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ &\leq I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\ &= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)] \\ &= I_{(O_1 \circ O_2)}(kl_1) \wedge I_{(O_1 \circ O_2)}(kl_2), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) &= F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \\ &\leq F_{O_1}(k) \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\ &= [F_{O_1}(k) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k) \vee F_{O_2}(l_2)] \\ &= F_{(O_1 \circ O_2)}(kl_1) \vee F_{(O_1 \circ O_2)}(kl_2), \end{aligned}$$

for $kl_1, kl_2 \in P_1 \circ P_2$.

Case 2: For $k \in P_2$, $l_1l_2 \in P_{1h}$

$$\begin{aligned} T_{(O_{1h} \circ O_{2h})}((l_1k)(l_2k)) &= T_{O_2}(k) \wedge T_{O_{1h}}(l_1l_2) \\ &\leq T_{O_2}(k) \wedge [T_{O_1}(l_1) \wedge T_{O_1}(l_2)] \\ &= [T_{O_2}(k) \wedge T_{O_1}(l_1)] \wedge [T_{O_2}(k) \wedge T_{O_1}(l_2)] \\ &= T_{(O_1 \circ O_2)}(l_1k) \wedge T_{(O_1 \circ O_2)}(l_2k), \end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\circ O_{2h})}((l_1k)(l_2k)) &= I_{O_2}(k)\wedge I_{O_{1h}}(l_1l_2)\\ &\leq I_{O_2}(k)\wedge [I_{O_1}(l_1)\wedge I_{O_1}(l_2)]\\ &= [I_{O_2}(k)\wedge I_{O_1}(l_1)]\wedge [I_{O_2}(k)\wedge I_{O_1}(l_2)]\\ &= I_{(O_1\circ O_2)}(l_1k)\wedge I_{(O_1\circ O_2)}(l_2k),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\circ O_{2h})}((l_1k)(l_2k)) &= F_{O_2}(k)\vee F_{O_{1h}}(l_1l_2)\\ &\leq F_{O_2}(k)\vee [F_{O_1}(l_1)\vee F_{O_1}(l_2)]\\ &= [F_{O_2}(k)\vee F_{O_1}(l_1)]\vee [F_{O_2}(k)\vee F_{O_1}(l_2)]\\ &= F_{(O_1\circ O_2)}(l_1k)\vee F_{(O_1\circ O_2)}(l_2k),\end{aligned}$$

for $l_1k, l_2k \in P_1\circ P_2$.

Case 3: For $k_1k_2 \in P_{1h}, l_1, l_2 \in P_2$ such that $l_1 \neq l_2$

$$\begin{aligned}T_{(O_{1h}\circ O_{2h})}((k_1l_1)(k_2l_2)) &= T_{O_{1h}}(k_1k_2)\wedge T_{O_2}(l_1)\wedge T_{O_2}(l_2)\\ &\leq [T_{O_1}(k_1)\wedge T_{O_1}(k_2)]\wedge T_{O_2}(l_1)\wedge T_{O_2}(l_2)\\ &= [T_{O_1}(k_1)\wedge T_{O_2}(l_1)]\wedge [T_{O_1}(k_2)\wedge T_{O_2}(l_2)]\\ &= T_{(O_1\circ O_2)}(k_1l_1)\wedge T_{(O_1\circ O_2)}(k_2l_2),\end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\circ O_{2h})}((k_1l_1)(k_2l_2)) &= I_{O_{1h}}(k_1k_2)\wedge I_{O_2}(l_1)\wedge I_{O_2}(l_2)\\ &\leq [I_{O_1}(k_1)\wedge I_{O_1}(k_2)]\wedge [I_{O_2}(l_1)\wedge I_{O_2}(l_2)]\\ &= [I_{O_1}(k_1)\wedge I_{O_2}(l_1)]\wedge [I_{O_1}(k_2)\wedge I_{O_2}(l_2)]\\ &= I_{(O_1\circ O_2)}(k_1l_1)\wedge I_{(O_1\circ O_2)}(k_2l_2),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\circ O_{2h})}((k_1l_1)(k_2l_2)) &= F_{O_{1h}}(k_1k_2)\vee F_{O_2}(l_1)\vee F_{O_2}(l_2)\\ &\leq [F_{O_1}(k_1)\vee F_{O_1}(k_2)]\vee [F_{O_2}(l_1)\vee F_{O_2}(l_2)]\\ &= [F_{O_1}(k_1)\vee F_{O_2}(l_1)]\vee [F_{O_1}(k_2)\vee F_{O_2}(l_2)]\\ &= F_{(O_1\circ O_2)}(k_1l_1)\vee F_{(O_1\circ O_2)}(k_2l_2),\end{aligned}$$

for $k_1l_1, k_2l_2 \in P_1\circ P_2$.

All cases holds for $h = 1, 2, \ldots, r$. This completes the proof. □

Definition 2.28. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. The union of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1}\cup\check{G}_{i2} = (O_1\cup O_2, O_{11}\cup O_{21}, O_{12}\cup O_{22}, \ldots, O_{1r}\cup O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1\cup O_2)}(k) = (T_{O_1}\cup T_{O_2})(k) = T_{O_1}(k)\vee T_{O_2}(k)\\ I_{(O_1\cup O_2)}(k) = (I_{O_1}\cup I_{O_2})(k) = I_{O_1}(k)\vee I_{O_2}(k)\\ F_{(O_1\cup O_2)}(k) = (F_{O_1}\cup F_{O_2})(k) = F_{O_1}(k)\wedge F_{O_2}(k)\end{cases}$

for all $k \in P_1\cup P_2$,

(ii) $\begin{cases} T_{(O_{1h}\cup O_{2h})}(kl) = (T_{O_{1h}}\cup T_{O_{2h}})(kl) = T_{O_{1h}}(kl)\vee T_{O_{2h}}(kl)\\ I_{(O_{1h}\cup O_{2h})}(kl) = (I_{O_{1h}}\cup I_{O_{2h}})(kl) = I_{O_{1h}}(kl)\vee I_{O_{2h}}(kl)\\ F_{(O_{1h}\cup O_{2h})}(kl) = (F_{O_{1h}}\cup F_{O_{2h}})(kl) = F_{O_{1h}}(kl)\wedge F_{O_{2h}}(kl)\end{cases}$

for all $kl \in P_{1h}\cup P_{2h}$.

Example 2.29. The union of two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as

$$\check{G}_{i1} \cup \check{G}_{i2} = \{O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22}\}$$

and is represented in Fig. 8.

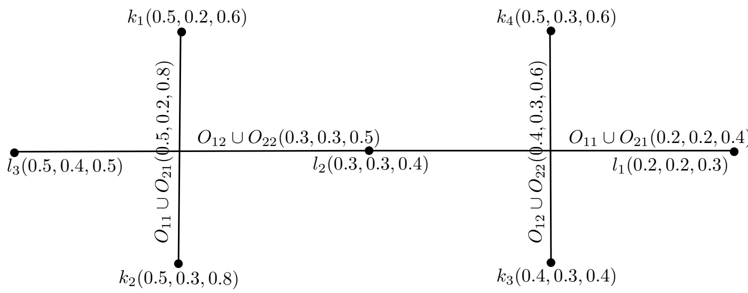


FIGURE 8. $\check{G}_{i1} \cup \check{G}_{i2}$

Theorem 2.30. *The union* $\check{G}_{i1} \cup \check{G}_{i2} = (O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22}, \ldots, O_{1r} \cup O_{2r})$ *of two INGSs of the GSs* $\check{G}_1$ *and* $\check{G}_2$ *is an INGS of* $\check{G}_1 \cup \check{G}_2$.

Proof. Let $k_1k_2 \in P_{1h} \cup P_{2h}$. There are two cases:

Case 1: For $k_1, k_2 \in P_1$, by definition 2.28, $T_{O_2}(k_1) = T_{O_2}(k_2) = T_{O_{2h}}(k_1k_2) = 0$, $I_{O_2}(k_1) = I_{O_2}(k_2) = I_{O_{2h}}(k_1k_2) = 0$, $F_{O_2}(k_1) = F_{O_2}(k_2) = F_{O_{2h}}(k_1k_2) = 1$. Thus,

$$\begin{aligned}
T_{(O_{1h} \cup O_{2h})}(k_1k_2) &= T_{O_{1h}}(k_1k_2) \vee T_{O_{2h}}(k_1k_2) \\
&= T_{O_{1h}}(k_1k_2) \vee 0 \\
&\le [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \vee 0 \\
&= [T_{O_1}(k_1) \vee 0] \wedge [T_{O_1}(k_2) \vee 0] \\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)] \\
&= T_{(O_1 \cup O_2)}(k_1) \wedge T_{(O_1 \cup O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h} \cup O_{2h})}(k_1k_2) &= I_{O_{1h}}(k_1k_2) \vee I_{Q_{2h}}(k_1k_2) \\
&= I_{O_{1h}}(k_1k_2) \vee 0 \\
&\le [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \vee 0 \\
&= [I_{O_1}(k_1) \vee 0] \wedge [I_{O_1}(k_2) \vee 0] \\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)] \\
&= I_{(O_1 \cup O_2)}(k_1) \wedge I_{(O_1 \cup O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\cup O_{2h})}(k_1k_2) &= F_{O_{1h}}(k_1k_2)\wedge F_{O_{2h}}(k_1k_2)\\ &= F_{O_{1i}}(k_1k_2)\wedge 1\\ &\leq [F_{O_1}(k_1)\vee F_{O_1}(k_2)]\wedge 1\\ &= [F_{O_1}(k_1)\wedge 1]\vee [F_{O_1}(k_2)\wedge 1]\\ &= [F_{O_1}(k_1)\wedge F_{O_2}(k_1)]\vee [F_{O_1}(k_2)\wedge F_{O_2}(k_2)]\\ &= F_{(O_1\cup O_2)}(k_1)\vee F_{(O_1\cup O_2)}(k_2),\end{aligned}$$

for $k_1, k_2 \in P_1 \cup P_2$.

Case 2: For $k_1, k_2 \in P_2$, by definition 2.28, $T_{O_1}(k_1) = T_{O_1}(k_2) = T_{O_{1h}}(k_1k_2) = 0$, $I_{O_1}(k_1) = I_{O_1}(k_2) = I_{O_{1h}}(k_1k_2) = 0$, $F_{O_1}(k_1) = F_{O_1}(q_2) = F_{O_{1h}}(k_1k_2) = 1$, so

$$\begin{aligned}T_{(O_{1h}\cup O_{2h})}(k_1k_2) &= T_{O_{1h}}(k_1k_2)\vee T_{O_{2i}}(k_1k_2)\\ &= T_{O_{2i}}(k_1k_2)\vee 0\\ &\leq [T_{O_2}(k_1)\wedge T_{O_2}(k_2)]\vee 0\\ &= [T_{O_2}(k_1)\vee 0]\wedge [T_{O_2}(k_2)\vee 0]\\ &= [T_{O_1}(k_1)\vee T_{O_2}(k_1)]\wedge [T_{O_1}(k_2)\vee T_{O_2}(k_2)]\\ &= T_{(O_1\cup O_2)}(k_1)\wedge T_{(O_1\cup O_2)}(k_2),\end{aligned}$$

$$\begin{aligned}I_{(O_{1h}\cup O_{2h})}(q_1k_2) &= I_{O_{1h}}(k_1k_2)\vee I_{O_{2h}}(k_1k_2)\\ &= I_{O_{2h}}(k_1k_2)\vee 0\\ &\leq [I_{O_2}(k_1)\wedge I_{O_2}(k_2)]\vee 0\\ &= [I_{O_2}(k_1)\vee 0]\wedge [I_{O_2}(k_2)\vee 0]\\ &= [I_{O_1}(k_1)\vee I_{O_2}(k_1)]\wedge [I_{O_1}(k_2)\vee I_{O_2}(k_2)]\\ &= I_{(O_1\cup O_2)}(k_1)\wedge I_{(O_1\cup O_2)}(k_2),\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}\cup O_{2h})}(k_1k_2) &= F_{O_{1h}}(k_1k_2)\wedge F_{O_{2h}}(k_1k_2)\\ &= F_{O_{2h}}(k_1k_2)\wedge 1\\ &\leq [F_{O_2}(k_1)\vee F_{O_2}(k_2)]\wedge 1\\ &= [F_{O_2}(k_1)\wedge 1]\vee [F_{O_2}(k_2)\wedge 1]\\ &= [F_{O_1}(k_1)\wedge F_{O_2}(k_1)]\vee [F_{O_1}(k_2)\wedge F_{O_2}(k_2)]\\ &= F_{(O_1\cup O_2)}(k_1)\vee F_{(O_1\cup O_2)}(k_2),\end{aligned}$$

for $k_1, k_2 \in P_1 \cup P_2$.

Both cases hold $\forall h \in \{1, 2, \dots, r\}$. This completes the proof. □

Theorem 2.31. *Let $\check{G} = (P_1 \cup P_2, P_{11} \cup P_{21}, P_{12} \cup P_{22}, \dots, P_{1r} \cup P_{2r})$ be the union of two GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, ..., P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, ..., P_{2r})$. Then every INGS $\check{G}_i = (O, O_1, O_2, ..., O_r)$ of $\check{G}$ is union of the two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ of GSs $\check{G}_1$ and $\check{G}_2$, respectively.*

Proof. Firstly, we define O_1, O_2, O_{1h} and O_{2h} for $h \in \{1, 2, \ldots, r\}$ as:
$T_{O_1}(k) = T_O(k), I_{O_1}(k) = I_O(k), F_{O_1}(k) = F_O(k)$, if $k \in P_1$,
$T_{O_2}(k) = T_O(k), I_{O_2}(k) = I_O(k), F_{O_2}(k) = F_O(k)$, if $k \in P_2$,
$T_{O_{1h}}(k_1k_2) = T_{O_h}(k_1k_2), I_{O_{1h}}(k_1k_2) = I_{O_h}(k_1k_2), F_{O_{1h}}(k_1k_2) = F_{O_h}(k_1k_2)$,
if $k_1k_2 \in P_{1h}$,
$T_{O_{2h}}(k_1k_2) = T_{O_h}(k_1k_2), I_{O_{2h}}(k_1k_2) = I_{O_h}(k_1k_2), F_{O_{2h}}(k_1k_2) = F_{O_h}(k_1k_2)$,
if $k_1k_2 \in P_{2h}$.
Then $O = O_1 \cup O_2$ and $O_h = O_{1h} \cup O_{2h}$, $h \in \{1, 2, \ldots, r\}$.
Now for $k_1k_2 \in P_{lh}$, $l = 1, 2$, $h = 1, 2, \ldots, r$,
$T_{O_{lh}}(k_1k_2) = T_{O_h}(k_1k_2) \leq T_O(k_1) \wedge T_O(k_2) = T_{O_l}(k_1) \wedge T_{O_l}(k_2)$,
$I_{O_{lh}}(k_1k_2) = I_{O_h}(k_1k_2) \leq I_O(k_1) \wedge I_O(k_2) = I_{O_l}(k_1) \wedge I_{O_l}(k_2)$,
$F_{O_{lh}}(k_1k_2) = F_{O_h}(k_1k_2) \leq F_O(k_1) \vee F_O(k_2) = F_{O_l}(k_1) \vee F_{O_l}(k_2)$, i.e.,
$\check{G}_{il} = (O_l, O_{l1}, O_{l2}, ..., O_{lr})$ is an INGS of $\check{G}_l$, l = 1,2.
Thus $\check{G}_i = (O, O_1, O_2, ..., O_r)$, an INGS of $\check{G} = \check{G}_1 \cup \check{G}_2$, is the union of the two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$. □

Definition 2.32. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be INGSs of GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively and let $P_1 \cap P_2 = \varnothing$. *Join* of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} + \check{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1+O_2)}(k) = T_{(O_1 \cup O_2)}(k) \\ I_{(O_1+O_2)}(k) = I_{(O_1 \cup O_2)}(k) \\ F_{(O_1+O_2)}(k) = F_{(O_1 \cup O_2)}(k) \end{cases}$
for all $k \in P_1 \cup P_2$,

(ii) $\begin{cases} T_{(O_{1h}+O_{2h})}(kl) = T_{(O_{1h} \cup O_{2h})}(kl) \\ I_{(O_{1h}+O_{2h})}(kl) = I_{(O_{1h} \cup O_{2h})}(kl) \\ F_{(O_{1h}+O_{2h})}(kl) = F_{(O_{1h} \cup O_{2h})}(kl) \end{cases}$
for all $kl \in P_{1h} \cup P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h}+O_{2h})}(kl) = (T_{O_{1h}} + T_{O_{2h}})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_{1h}+O_{2h})}(kl) = (I_{O_{1h}} + I_{O_{2h}})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_{1h}+O_{2h})}(kl) = (F_{O_{1h}} + F_{O_{2h}})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$
for all $k \in P_1$, $l \in P_2$.

Example 2.33. The join of two INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2 is defined as $\check{G}_{i1} + \check{G}_{i2} = \{O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}\}$ and is represented in the Fig. 9.

Theorem 2.34. *The join* $\check{G}_{i1} + \check{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r})$ *of two INGSs of GSs* $\check{G}_1$ *and* $\check{G}_2$ *is INGS of* $\check{G}_1 + \check{G}_2$*.*

Proof. Let $k_1k_2 \in P_{1h} + P_{2h}$. There are three cases:

Case 1: For $k_1, k_2 \in P_1$, by definition 2.32, $T_{O_2}(k_1) = T_{O_2}(k_2) = T_{O_{2h}}(k_1k_2) = 0$, $I_{O_2}(k_1) = I_{O_2}(k_2) = I_{O_{2h}}(k_1k_2) = 0$, $F_{O_2}(k_1) = F_{O_2}(k_2) = F_{O_{2h}}(k_1k_2) =$

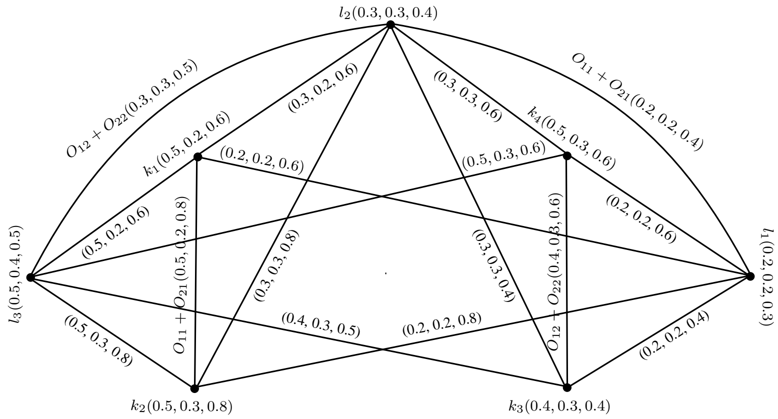


FIGURE 9. $\check{G}_{i1}+\check{G}_{i2}$

1, so,

$$\begin{aligned}
T_{(O_{1h}+O_{2h})}(k_1k_2) &= T_{O_{1h}}(k_1k_2) \vee T_{O_{2h}}(k_1k_2)\\
&= T_{O_{1h}}(k_1k_2) \vee 0\\
&\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \vee 0\\
&= [T_{O_1}(k_1) \vee 0] \wedge [T_{O_1}(q_2) \vee 0]\\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)]\\
&= T_{(O_1+O_2)}(k_1) \wedge T_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h}+O_{2h})}(k_1k_2) &= I_{O_{1h}}(k_1k_2) \vee I_{O_{2h}}(k_1k_2)\\
&= I_{O_{1h}}(k_1k_2) \vee 0\\
&\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \vee 0\\
&= [I_{O_1}(k_1) \vee 0] \wedge [I_{O_1}(k_2) \vee 0]\\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)]\\
&= I_{(O_1+O_2)}(k_1) \wedge I_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
F_{(O_{1h}+O_{2h})}(k_1k_2) &= F_{O_{1h}}(k_1k_2) \wedge F_{O_{2h}}(k_1k_2)\\
&= F_{O_{1h}}(k_1k_2) \wedge 1\\
&\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \wedge 1\\
&= [F_{O_1}(k_1) \wedge 1] \vee [F_{O_1}(k_2) \wedge 1]\\
&= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)]\\
&= F_{(O_1+O_2)}(k_1) \vee F_{(O_1+O_2)}(k_2),
\end{aligned}$$

for $k_1, k_2 \in P_1 + P_2$.

Case 2: For $k_1, k_2 \in P_2$, by definition 2.32, $T_{O_1}(k_1) = T_{O_1}(k_2) = T_{O_{1h}}(k_1k_2) = 0$, $I_{O_1}(k_1) = I_{O_1}(k_2) = I_{O_{1h}}(k_1k_2) = 0$, $F_{O_1}(k_1) = F_{O_1}(k_2) = F_{O_{1h}}(k_1k_2) = 1$, so

$$\begin{aligned}
T_{(O_{1h}+O_{2h})}(k_1k_2) &= T_{O_{1i}}(k_1k_2) \vee T_{O_{2i}}(k_1k_2)\\
&= T_{O_{2i}}(k_1k_2) \vee 0\\
&\leq [T_{O_2}(k_1) \wedge T_{O_2}(k_2)] \vee 0\\
&= [T_{O_2}(k_1) \vee 0] \wedge [T_{O_2}(k_2) \vee 0]\\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)]\\
&= T_{(O_1+O_2)}(k_1) \wedge T_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h}+O_{2h})}(k_1k_2) &= I_{O_{1h}}(k_1k_2) \vee I_{O_{2h}}(k_1k_2)\\
&= I_{O_{2h}}(k_1k_2) \vee 0\\
&\leq [I_{O_2}(k_1) \wedge I_{O_2}(k_2)] \vee 0\\
&= [I_{O_2}(k_1) \vee 0] \wedge [I_{O_2}(k_2) \vee 0]\\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)]\\
&= I_{(O_1+O_2)}(k_1) \wedge I_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
F_{(O_{1h}+O_{2h})}(k_1k_2) &= F_{O_{1h}}(k_1k_2) \wedge F_{O_{2h}}(k_1k_2)\\
&= F_{O_{2h}}(k_1k_2) \wedge 1\\
&\leq [F_{O_2}(k_1) \vee F_{O_2}(k_2)] \wedge 1\\
&= [F_{O_2}(k_1) \wedge 1] \vee [F_{O_2}(k_2) \wedge 1]\\
&= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)]\\
&= F_{(O_1+O_2)}(k_1) \vee F_{(O_1+O_2)}(k_2),
\end{aligned}$$

for $q_1, q_2 \in P_1 + P_2$.

Case 3: For $k_1 \in P_1$, $k_2 \in P_2$, by definition 2.32,
$T_{O_1}(k_2) = T_{O_2}(k_1) = 0$, $I_{O_1}(k_2) = I_{O_2}(k_1) = 0$, $F_{O_1}(k_2) = F_{O_2}(k_1) = 1$, so

$$\begin{aligned}
T_{(O_{1h}+O_{2h})}(k_1k_2) &= T_{O_1}(q_1) \wedge T_{O_2}(k_2)\\
&= [T_{O_1}(k_1) \vee 0] \wedge [T_{O_2}(k_2) \vee 0]\\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_2}(k_2) \vee T_{O_1}(k_2)]\\
&= T_{(O_1+O_2)}(k_1) \wedge T_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}
I_{(O_{1h}+O_{2h})}(k_1k_2) &= I_{O_1}(k_1) \wedge I_{O_2}(k_2)\\
&= [I_{O_1}(k_1) \vee 0] \wedge [I_{O_2}(k_2) \vee 0]\\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_2}(k_2) \vee I_{O_1}(k_2)]\\
&= I_{(O_1+O_2)}(k_1) \wedge I_{(O_1+O_2)}(k_2),
\end{aligned}$$

$$\begin{aligned}F_{(O_{1h}+O_{2h})}(k_1k_2) &= F_{O_1}(k_1) \vee F_{O_2}(k_2)\\ &= [F_{O_1}(k_1) \wedge 1] \vee [F_{O_2}(k_2) \wedge 1]\\ &= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_2}(k_2) \wedge F_{O_1}(k_2)]\\ &= F_{(O_1+O_2)}(k_1) \vee F_{(O_1+O_2)}(k_2),\end{aligned}$$

for $k_1, k_2 \in P_1 + P_2$.

All these cases hold $\forall h \in \{1, 2, \ldots, r\}$. This completes the proof. □

Theorem 2.35. *If $\check{G} = (P_1 + P_2, P_{11} + P_{21}, P_{12} + P_{22}, \ldots, P_{1r} + P_{2r})$ is the join of the two GSs $\check{G}_1 = (P_1, P_{11}, P_{12}, ..., P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, ..., P_{2r})$. Then each strong INGS $\check{G}_i = (O, O_1, O_2, ..., O_r)$ of $\check{G}$, is join of the two strong INGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ of GSs $\check{G}_1$ and $\check{G}_2$, respectively.*

Proof. We define O_l and O_{lh} for $l = 1, 2$ and $h = 1, 2, \ldots, r$ as:

$T_{O_l}(k) = T_O(k), I_{O_k}(k) = I_O(k), F_{O_l}(k) = F_O(k)$, if $k \in P_l$,

$T_{O_{lh}}(k_1k_2) = T_{O_h}(k_1k_2), I_{O_{lh}}(k_1k_2) = I_{O_h}(k_1k_2), F_{O_{lh}}(k_1k_2) = F_{O_h}(k_1k_2)$, if $k_1k_2 \in P_{lh}$.

Now for $k_1k_2 \in P_{lh}$, $l = 1, 2$, $h = 1, 2, \ldots, r$,

$T_{O_{lh}}(k_1k_2) = T_{O_h}(k_1k_2) = T_O(k_1) \wedge T_O(k_2) = T_{O_l}(k_1) \wedge T_{O_l}(k_2)$,

$I_{O_{lh}}(k_1k_2) = I_{O_h}(k_1k_2) = I_O(k_1) \wedge I_O(k_2) = I_{O_l}(k_1) \wedge I_{O_l}(k_2)$,

$F_{O_{lh}}(k_1k_2) = F_{O_h}(k_1k_2) = F_O(k_1) \vee F_O(k_2) = F_{O_l}(k_1) \vee F_{O_l}(k_2)$, i.e.,

$\check{G}_{il} = (O_l, O_{l1}, O_{l2}, ..., O_{lr})$ is strong INGS of $\check{G}_l$, $l = 1, 2$.

Moreover, $\check{G}_i$ is the join of $\check{G}_{i1}$ and $\check{G}_{i2}$ as shown:

According to the definitions 2.28 and 2.32, $O = O_1 \cup O_2 = O_1 + O_2$ and $O_h = O_{1h} \cup O_{2h} = O_{1h} + O_{2h}$, $\forall k_1k_2 \in P_{1h} \cup P_{2h}$.

When $k_1k_2 \in P_{1h} + P_{2h}$ $(P_{1h} \cup P_{2h})$, i.e., $k_1 \in P_1$ and $k_2 \in P_2$,

$T_{O_h}(k_1k_2) = T_O(k_1) \wedge T_O(k_2) = T_{O_l}(k_1) \wedge T_{O_l}(k_2) = T_{(O_{1h}+O_{2h})}(k_1k_2)$,

$I_{O_h}(k_1k_2) = I_O(k_1) \wedge I_O(k_2) = I_{O_l}(k_1) \wedge I_{O_l}(k_2) = I_{(O_{1h}+O_{2h})}(k_1k_2)$, $F_{O_h}(k_1k_2) = F_O(k_1) \vee F_O(k_2) = F_{O_l}(k_1) \vee F_{O_l}(k_2) = F_{(O_{1h}+O_{2h})}(k_1k_2)$,

when $k_1 \in P_2$, $k_2 \in P_1$, we get similar calculations. It's true for $h = 1, 2, \ldots, r$. This completes the proof. □

3. **Application**

According to IMF data, 1.75 billion people are living in poverty, their living is estimated to be less than two dollars a day. Poverty changes by region, for example in Europe it is 3%, and in the Sub-Saharan Africa it is up to 65%. We rank the countries of the World as poor or rich, using their GDP per capita as scale. Poor countries are trying to catch up with rich or developed countries. But this ratio is very small, that's why trade of poor countries among themselves is very important. There are different types of trade among poor countries, for example: agricultural or food items, raw minerals, medicines, textile materials, industrials goods etc. Using INGS, we can estimate between any two poor countries which trade is comparatively stronger than others. Moreover, we can decide(judge) which country has large number of resources for particular type of goods and better circumstances for its trade. We can figure out, for which trade, an external investor can invest his money in these poor countries. Further, it will be easy to judge that in which field these poor countries are trying to

TABLE 1. IN set O of nine poor countries on globe

Poor Country	T	I	F
Congo	0.5	0.3	0.2
Liberia	0.4	0.4	0.3
Burundi	0.4	0.4	0.4
Tanzania	0.5	0.5	0.4
Uganda	0.4	0.4	0.5
Sierra Leone	0.5	0.4	0.4
Zimbabwe	0.3	0.4	0.4
Kenya	0.5	0.3	0.3
Zambia	0.4	0.4	0.4

be better, and can be helped. It will also help in deciding that in which trade they are weak, and should be facilitated, so that they can be independent and improve their living standards.
We consider a set of nine poor countries in the World:

$$P = \{\text{Congo}, \text{Liberia}, \text{Burundi}, \text{Tanzania}, \text{Ugenda}, \text{SierriaLeone}, \text{Zimbabwe}, \text{Kenya}, \text{Zambia}\}.$$

Let O be the IN set on P, as defined in Table 1. In Table 1, symbol T demonstrates the positive aspects of that poor country, symbol I indicates its negative aspects, whereas F denotes the percentage of ambiguity of its problems for the World. Let we use following alphabets for country names:
CO = Congo, L = Liberia, B = Burundi, T = Tanzania, U = Uganda, SL = Sierra Leone, ZI = Zimbabwe, K = Kenya, ZA = Zambia. For every pair of poor countries in set P, different trades with their T, I and F values are demonstrated in Tables $2-8$, where T, F and I indicates the percentage of occurrence, non-occurrence and uncertainty, respectively of a particular trade between those two poor countries

TABLE 2. IN set of different types of trade between Congo and other poor countries in P

Type of trade	(CO, L)	(CO, B)	(CO, T)	(CO, U)	(CO, K)
Food items	(0.1, 0.2, 0.3)	(0.4, 0.2, 0.1)	(0.2, 0.1, 0.4)	(0.4, 0.3, 0.5)	(0.2, 0.1, 0.3)
Chemicals	(0.2, 0.4, 0.3)	(0.1, 0.2, 0.1)	(0.1, 0.2, 0.4)	(0.3, 0.2, 0.4)	(0.5, 0.1, 0.1)
Oil	(0.4, 0.2, 0.1)	(0.4, 0.3, 0.2)	(0.5, 0.1, 0.2)	(0.4, 0.2, 0.2)	(0.5, 0.3, 0.1)
Raw minerals	(0.3, 0.1, 0.1)	(0.4, 0.3, 0.3)	(0.4, 0.2, 0.2)	(0.4, 0.1, 0.2)	(0.5, 0.1, 0.1)
Textile products	(0.2, 0.3, 0.3)	(0.1, 0.3, 0.4)	(0.1, 0.2, 0.4)	(0.1, 0.3, 0.2)	(0.2, 0.1, 0.3)
Gold and diamonds	(0.4, 0.1, 0.1)	(0.4, 0.2, 0.2)	(0.2, 0.2, 0.4)	(0.2, 0.2, 0.4)	(0.1, 0.3, 0.3)

TABLE 3. IN set of different types of trade between Liberia and other poor countries in P

Type of trade	(L, B)	(L, T)	(L, U)	(L, SL)	(L, ZI)
Food items	(0.4, 0.2, 0.2)	(0.4, 0.3, 0.2)	(0.3, 0.3, 0.4)	(0.3, 0.3, 0.3)	(0.2, 0.3, 0.3)
Chemicals	(0.2, 0.2, 0.4)	(0.1, 0.4, 0.3)	(0.3, 0.3, 0.3)	(0.2, 0.2, 0.4)	(0.1, 0.3, 0.3)
Oil	(0.1, 0.1, 0.4)	(0.2, 0.3, 0.3)	(0.1, 0.1, 0.4)	(0.2, 0.4, 0.3)	(0.2, 0.2, 0.3)
Raw minerals	(0.3, 0.1, 0.3)	(0.2, 0.2, 0.3)	(0.2, 0.1, 0.4)	(0.3, 0.2, 0.3)	(0.2, 0.1, 0.3)
Textile products	(0.1, 0.3, 0.4)	(0.1, 0.3, 0.3)	(0.2, 0.1, 0.3)	(0.1, 0.2, 0.3)	(0.2, 0.2, 0.3)
Gold and diamonds	(0.2, 0.1, 0.4)	(0.2, 0.1, 0.3)	(0.3, 0.1, 0.3)	(0.4, 0.1, 0.1)	(0.3, 0.1, 0.1)

TABLE 4. IN set of different types of trade between Burundi and other poor countries in P

Type of trade	(B, T)	(B, U)	(B, SL)	(B, ZI)	(B, K)
Food items	(0.3, 0.2, 0.2)	(0.4, 0.1, 0.2)	(0.3, 0.3, 0.1)	(0.3, 0.3, 0.2)	(0.3, 0.2, 0.2)
Chemicals	(0.1, 0.2, 0.3)	(0.2, 0.1, 0.3)	(0.2, 0.4, 0.3)	(0.3, 0.4, 0.3)	(0.3, 0.3, 0.1)
Oil	(0.1, 0.1, 0.4)	(0.2, 0.3, 0.4)	(0.2, 0.4, 0.3)	(0.2, 0.2, 0.5)	(0.1, 0.3, 0.4)
Raw minerals	(0.2, 0.1, 0.3)	(0.4, 0.2, 0.3)	(0.4, 0.2, 0.4)	(0.3, 0.2, 0.2)	(0.4, 0.2, 0.2)
Textile products	(0.3, 0.1, 0.1)	(0.2, 0.4, 0.3)	(0.3, 0.2, 0.2)	(0.3, 0.2, 0.1)	(0.4, 0.1, 0.2)
Gold and diamonds	(0.3, 0.2, 0.3)	(0.3, 0.4, 0.3)	(0.1, 0.4, 0.2)	(0.2, 0.4, 0.2)	(0.2, 0.3, 0.4)

TABLE 5. IN set of different types of trade between Tanzania and other poor countries in P

Type of trade	(T, U)	(T, SL)	(T, ZI)	(T, K)	(T,ZA)
Food items	(0.4, 0.2, 0.1)	(0.5, 0.1, 0.1)	(0.3, 0.1, 0.2)	(0.4, 0.3, 0.2)	(0.3, 0.2, 0.2)
Chemicals	(0.2, 0.3, 0.3)	(0.2, 0.3, 0.4)	(0.2, 0.3, 0.3)	(0.4, 0.1, 0.4)	(0.3, 0.4, 0.4)
Oil	(0.1, 0.3, 0.3)	(0.4, 0.1, 0.3)	(0.3, 0.4, 0.2)	(0.2, 0.3, 0.3)	(0.1, 0.3, 0.3)
Raw minerals	(0.3, 0.3, 0.4)	(0.4, 0.3, 0.3)	(0.3, 0.2, 0.1)	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.3)
Textile products	(0.2, 0.4, 0.3)	(0.2, 0.4, 0.4)	(0.1, 0.3, 0.4)	(0.2, 0.3, 0.2)	(0.4, 0.1, 0.2)
Gold and diamonds	(0.3, 0.4, 0.3)	(0.4, 0.3, 0.4)	(0.3, 0.1, 0.1)	(0.2, 0.2, 0.2)	(0.4, 0.3, 0.3)

TABLE 6. IN set of different types of trade between Sierra Leone and other poor countries in P

Type of trade	(SL, ZI)	(SL, K)	(SL, ZA)	(SL, CO)	(L, K)
Food items	(0.3, 0.3, 0.2)	(0.4, 0.2, 0.1)	(0.2, 0.4, 0.3)	(0.5, 0.1, 0.1)	(0.4, 0.1, 0.2)
Chemicals	(0.2, 0.3, 0.4)	(0.3, 0.2, 0.2)	(0.2, 0.4, 0.4)	(0.2, 0.2, 0.3)	(0.2, 0.3, 0.3)
Oil	(0.1, 0.3, 0.4)	(0.2, 0.2, 0.3)	(0.3, 0.4, 0.2)	(0.5, 0.2, 0.1)	(0.3, 0.3, 0.3)
Raw minerals	(0.3, 0.2, 0.2)	(0.5, 0.2, 0.1)	(0.3, 0.1, 0.1)	(0.3, 0.3, 0.3)	(0.4, 0.1, 0.2)
Textile products	(0.2, 0.4, 0.2)	(0.3, 0.2, 0.3)	(0.2, 0.2, 0.4)	(0.2, 0.2, 0.3)	(0.3, 0.3, 0.2)
Gold and diamonds	(0.3, 0.1, 0.1)	(0.1, 0.2, 0.4)	(0.2, 0.3, 0.3)	(0.4, 0.1, 0.2)	(0.3, 0.2, 0.3)

Table 7. IN set of different types of trade between Zimbabwe and other poor countries in P

Type of trade	(ZI, K)	(ZI, ZA)	(ZI, U)	(ZI, CO)
Food items	(0.3, 0.2, 0.2)	(0.3, 0.1, 0.1)	(0.3, 0.1, 0.1)	(0.2, 0.1, 0.1)
Chemicals	(0.3, 0.3, 0.2)	(0.2, 0.4, 0.3)	(0.3, 0.2, 0.2)	(0.2, 0.1, 0.2)
Oil	(0.1, 0.3, 0.3)	(0.1, 0.4, 0.4)	(0.3, 0.2, 0.1)	(0.3, 0.1, 0.1)
Raw minerals	(0.3, 0.1, 0.2)	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.3)	(0.2, 0.3, 0.1)
Textile products	(0.2, 0.2, 0.2)	(0.2, 0.4, 0.3)	(0.2, 0.3, 0.3)	(0.2, 0.3, 0.1)
Gold and diamonds	(0.3, 0.3, 0.1)	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.2)	(0.3, 0.2, 0.1)

Table 8. IN set of different types of trade between Zambia and other poor countries in P

Type of trade	(ZA, CO)	(ZA, L)	(ZA, B)	(ZA, K)
Food items	(0.3, 0.1, 0.2)	(0.3, 0.1, 0.2)	(0.4, 0.2, 0.1)	(0.3, 0.1, 0.3)
Chemicals	(0.2, 0.2, 0.2)	(0.2, 0.2, 0.1)	(0.3, 0.2, 0.2)	(0.3, 0.1, 0.1)
Oil	(0.4, 0.1, 0.1)	(0.2, 0.1, 0.1)	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.2)
Raw minerals	(0.3, 0.1, 0.1)	(0.4, 0.1, 0.1)	(0.4, 0.2, 0.2)	(0.4, 0.1, 0.1)
Textile products	(0.2, 0.2, 0.2)	(0.2, 0.2, 0.3)	(0.2, 0.3, 0.2)	(0.3, 0.1, 0.2)
Gold and diamonds	(0.1, 0.2, 0.4)	(0.4, 0.3, 0.2)	(0.2, 0.3, 0.2)	(0.3, 0.2, 0.1)

Many relations can be defined on the set P, we define following relations on set P as:
P_1 = Food items, P_2 = Chemicals, P_3 = Oil, P_4 = Raw minerals, P_5 = Textile products, P_6 = Gold and diamonds, such that $(P, P_1, P_2, P_3, P_4, P_5, P_6)$ is a GS. Any element of a relation demonstrates a particular trade between those two poor countries. As $(P, P_1, P_2, P_3, P_4, P_5, P_6)$ is GS, that's why any element can appear in only one relation. Therefore, any element will be considered in that relation, whose value of T is high, and values of I, F are comparatively low, using data of above tables.
Write down T, I and F values of the elements in relations according to above data, such that O_1, O_2, O_3, O_4, O_5, O_6 are IN sets on relations P_1, P_2, P_3, P_4, P_5, P_6, respectively.
Let $P_1 = \{(\text{Burundi}, \text{Congo}), (\text{SierraLeone}, \text{Congo}), (\text{Burundi}, \text{Zambia})\}$, $P_2 = \{(Kenya, Congo)\}$,
$P_3 = \{(\text{Congo}, \text{Zambia}), (\text{Congo}, \text{Tanzania}), (\text{Zimbabwe}, \text{Congo})\}$,
$P_4 = \{(\text{Congo}, \text{Uganda}), (\text{SierraLeone}, \text{Kenya}), (\text{Zambia}, \text{Kenya})\}$,
$P_5 = \{(\text{Burundi}, \text{Zimbabwe}), (\text{Tanzania}, \text{Burundi})\}$,
$P_6 = \{(\text{SierraLeone}, \text{Liberia}), (\text{Uganda}, \text{SierraLeone}), (\text{Zimbabwe}, \text{SierraLeone})\}$.
Let $O_1 = \{((B, CO), 0.4, 0.2, 0.1), ((SL, CO), 0.5, 0.1, 0.1), ((B, ZA), 0.4, 0.2, 0.1)\}$,
$O_2 = \{((K, CO), 0.5, 0.1, 0.1)\}$, $O_3 = \{((CO, ZA), 0.4, 0.1, 0.1), ((CO, T), 0.5, 0.1, 0.2),$
$((ZI, CO), 0.3, 0.1, 0.1)\}$, $O_4 = \{((CO, U), 0.4, 0.1, 0.2), ((SL, K), 0.5, 0.2, 0.1), ((ZA, K), 0.4, 0.1, 0.1)\}$,
$O_5 = \{((B, ZI), 0.3, 0.2, 0.1), ((T, B), 0.3, 0.1, 0.1)\}$, $O_6 = \{((SL, L), 0.4, 0.1, 0.1), ((U, SL), 0.4, 0.2, 0.1),$
$((ZI, SL), 0.3, 0.1, 0.1)\}$. Obviously, $(O, O_1, O_2, O_3, O_4, O_5, O_6)$ is an INGS as shown in Fig. 10.

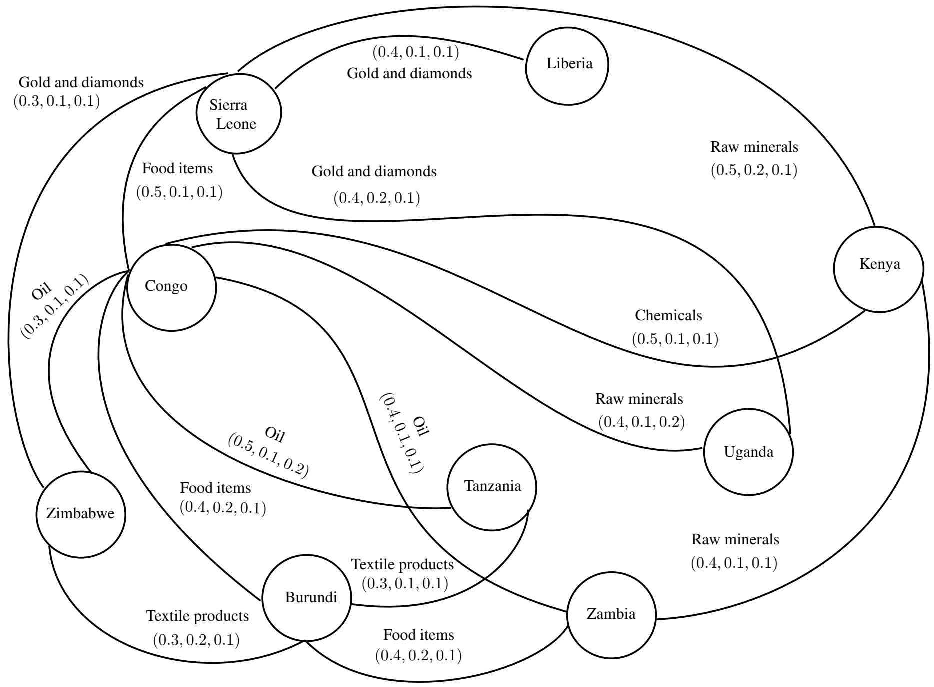


FIGURE 10. INGS indicating eminent trade between any two poor countries

Every edge of this INGS demonstrates the prominent trade between two poor countries, for example prominent trade between Congo and Zambia is Oil, its T, F and I values are 0.4, 0.1 and 0.1, respectively. According to these values, despite of poverty, circumstances of Congo and Zambia are 40% favorable for oil trade, 10% are unfavorable, and 10% are uncertain, that is, sometimes they may be favorable and sometimes unfavorable. We can observe that Congo is vertex with highest vertex degree for relation oil and Sierra Leone is vertex with highest vertex degree for relation gold and diamonds. That is, among these nine poor countries, Congo is most favorable for oil trade, and Sierra Leone is most favorable for trade of gold and diamonds. This INGS will be useful for those investors, who are interested to invest in these nine poor countries. For example an investor can invest in oil in Congo. And if someone wants to invest in gold and diamonds, this INGS will help him that Sierra Leone is most favorable.

A big advantage of this INGS is that United Nations, IMF, World Bank, and rich countries can be aware of the fact that in which fields of trade, these poor countries are trying to be better and can be helped to make their economic conditions better. Moreover, INGS of poor countries can be very beneficial for them, it may increase trade as well as foreign aid and economic help from the World, and can present their better aspects before the World.

We now explain general procedure of this application by following algorithm.
Algorithm:

1. Input a vertex set $P = \{C_1, C_2, \ldots, C_n\}$ and a IN set O defined on set P.
2. Input IN set of trade of any vertex with all other vertices and calculate T, F, and I of each pair of vertices using, $T(C_iC_j) \leq \min(T(C_i), T(C_j))$, $F(C_iC_j) \leq \max(F(C_i), F(C_j))$, $I(C_iC_j) \leq \min(I(C_i), I(C_j))$.
3. Repeat Step 2 for each vertex in set P.
4. Define relations $P_1, P_2, \ldots, P_n$ on the set P such that $(P, P_1, P_2, \ldots, P_n)$ is a GS.
5. Consider an element of that relation, for which its value of T is comparatively high, and its values of F and I are low than other relations.
6. Write down all elements in relations with T, F and I values, corresponding relations $O_1, O_2, \ldots, O_n$ are IN sets on $P_1, P_2, P_3, \ldots, P_n$, respectively and $(O, O_1, O_2, \ldots, O_n)$ is an INGS.

4. Conclusions

Fuzzy graphical models are highly utilized in applications of computer science. Especially in database theory, cluster analysis, image capturing, data mining, control theory, neural networks, expert systems and artificial intelligence. In this research paper, we have introduced certain operations on intuitionistic neutrosophic graph structures. We have discussed a novel and worthwhile real-life application of intuitionistic neutrosophic graph structure in decision-making. We have intensions to generalize our concepts to (1) Applications of IN soft GSs in decision-making (2) Applications of IN rough fuzzy GSs in decision-making, (3) Applications of IN fuzzy soft GSs in decision-making, and (4) Applications of IN rough fuzzy soft GSs in decision-making.

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References

[1] M. Akram and R. Akmal, Application of bipolar fuzzy sets in graph structures, Applied Computational Intelligence and Soft Computing (2016) 13 pages.

[2] M. Akram and S. Shahzadi, Neutrosophic soft graphs with application, Journal of Intelligent & Fuzzy Systems 32 (2017) 841–858.

[3] M. Akram and S. Shahzadi, Representation of graphs using intuitionistic neutrosophic soft sets, Journal of Mathematical Analysis 7 (6) (2016) 31–53.

[4] M. Akram and G. Shahzadi, Operations on single-valued neutrosophic graphs, Journal of Uncertain System 11 (2017) 1–26.

[5] M. Akram, S. Shahzadi, and A. Borumand saeid, Single-valued neutrosophic hypergraphs, TWMS Journal of Applied and Engineering Mathematics, 2016 (In press).

[6] M. Akram and M. Sitara, Novel applications of single-valued neutrosophic graph structures in decision making, Journal of Applied Mathematics and Computing, 2016, DOI 10.1007/s12190-017-1084-5 1–32.

[7] M. Akram and A. Adeel, Representation of labeling tree based on m- polar fuzzy sets, Ann. Fuzzy Math. Inform. 13 (2) (2017) 1–9.

[8] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[9] P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognition Letters 6 (5) (1987) 297–302.
[10] M. Bhowmik and M. Pal, Intuitionistic neutrosophic set, Journal of Information and Computing Science 4 (2) (2009) 142–152.
[11] M. Bhowmik and M. Pal, Intuitionistic neutrosophic set relations and some of its properties, Journal of Information and Computing Science 5 (3) 183–192.
[12] K. R. Bhutani and A. Rosenfeld, Strong arcs in fuzzy graphs, Inform. Sci. 152 (2003) 319–326.
[13] S. Broumi, M. Talea, A. Bakali, and F. Smarandache, Single-valued neutrosophic graphs, Journal of New Theory 10 (2016) 86–101.
[14] T. Dinesh, A study on graph structures, incidence algebras and their fuzzy analogues [Ph.D.thesis], Kannur University, Kannur India 2011.
[15] T. Dinesh and T. V. Ramakrishnan, On generalised fuzzy graph structures, Applied Mathematical Sciences 5 (4) (2011) 173–180.
[16] A. Kauffman, Introduction a la Theorie des Sous-emsembles Flous, Masson et Cie Vol.1 1973.
[17] M. G. Karunambigai and R. Buvaneswari, Degrees in intuitionistic fuzzy graphs, Ann. Fuzzy Math. Inform. 13 (2017) 1–13.
[18] J. N. Mordeson and P. Chang-Shyh, Operations on fuzzy graphs, Inform. Sci. 79 (1994) 159–170.
[19] T.K. Mondal and S.K. Samanta, Generalized intuitionistic fuzzy sets, J. Fuzzy Math. 10 (4) (2002) 839–862.
[20] J. J. Peng, J. Q. Wang, H. Y. Zhang and X. H. Chen, An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets, Applied Soft Computing 25 (2014) 336–346.
[21] A. Rosenfeld, Fuzzy graphs, Fuzzy Sets and their Applications(L.A.Zadeh, K.S.Fu, M.Shimura, Eds.), Academic Press, New York (1975) 77–95.
[22] N. Shah and A. Hussain, Neutrosophic soft graphs, Neutrosophic Set and Systems 11 (2016) 31–44.
[23] E. Sampathkumar, Generalized graph structures, Bulletin of Kerala Mathematics Association 3 (2) (2006) 65–123.
[24] F. Smarandache, Neutrosophy Neutrosophic Probability, Set, and Logic, Amer Res Press Rehoboth USA 1998.
[25] F. Smarandache, A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth 1999.
[26] I. Turksen, Interval-valued fuzzy sets based on normal form, Fuzzy Sets and Systems 20 (1986) 191–210.
[27] H. Wang, F. Smarandache, Y. Q. Zhang and R. Sunderraman, Single valued neutrosophic sets, Multispace and Multistructure 4 (2010) 410-413.
[28] J. Ye, Single-valued neutrosophic minimum spanning tree and its clustering method, Journal of Intelligent Systems 23 (3) (2014) 311–324.
[29] L. A. Zadeh, Fuzzy sets, Information and control 8 (1965) 338–353.
[30] L. A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.

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